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Multi-mode models of flow and of solute dispersion in shallow water. Part 3. Horizontal dispersion tensor for velocity

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Svendsen & Putrevu (1994) revealed that the much larger off-shore than vertical effective viscosity for longshore currents is the consequence of a shear dispersion mechanism. The multi-mode representation for the flow gives a mathematical framework within which a more general derivation can be made. It is shown that to a first approximation the horizontal shear dispersion tensor for velocity is the same as that for solutes.

1. Introduction

It is an important breakthrough when a large empirical term in a widely used model equation is found to be a derivable consequence of the underlying physics. This is what Svendsen & Putrevu (1994) have done for the shallow water (vertically averaged) momentum equations. They were concerned that to reproduce the observed longshore currents, the necessary off-shore eddy viscosities are factors of 35 to 70 times larger than the vertical eddy viscosities needed to reproduce the vertical structure of the flow. A similar disparity for solute spreading had been explained by Elder (1959) as an instance of the Taylor (1953) shear dispersion mechanism. Svendsen & Putrevu (1994) showed that for longshore currents there is indeed a shear dispersion mechanism which gives an effective off-shore eddy viscosity of a suitably large magnitude.

An apparently idiosyncratic aspect of the flow model used by Svendsen & Putrevu (1994) is the allowance for a non-zero slip velocity at the bed. In their calculations without the slip there is zero vertical shear, zero off-shore eddy viscosity and the agreement with observations is lost. One purpose of the present calculations is to re-derive the results obtained by Svendsen & Putrevu (1994) in the context of a more conventional flow model with zero flow at the bed.

Maron (1978) and Yu & Chang (1994) showed that for solute concentrations the Taylor (1953) shear dispersion model can be interpreted as being an accurate single-mode reduction, in a frame of reference moving at the speed appropriate to that mode, of a multi-mode representation for the concentration. The duality between multi-mode models of flow and of solute concentrations has been the theme of the present sequence of papers (Smith 1995*a*,*b*, referred to as Parts 1 and 2). In this Part 3, a moving-frame Taylor-type single-mode reduction is derived for the velocity components. The resulting model equations are a two-dimensional extension of the work of Svendsen & Putrevu (1994). A nice outcome is that to a first approximation, empirical horizontal eddy viscosities for the two-dimensional vector horizontal velocities should be replaced by the shear dispersion tensor for the scalar concentration of solutes.

2. Velocity modes

The vertical coordinate σ is the fractional height between the bed ($z = -h$) and the water surface ($z = \zeta$):

$$\sigma = \frac{z+h}{H} \quad \text{with} \quad H = h + \zeta. \tag{2.1a,b}$$

As in Svendsen & Putrevu (1994) and in Parts 1 and 2, we assume that the vertical exchange of momentum can be approximated in terms of a reference eddy viscosity

$$\nu(x, y, \sigma, t) = N(x, y, t)\hat{\nu}(\sigma) + \nu'. \tag{2.2}$$

N gives the magnitude and $\hat{\nu}$ the shape of the reference eddy viscosity. The selection of N or ν' to minimize error is given later in equation (3.3). The work of Jansons & Rogers (1995) illustrates that despite the stochastic character of the vertical mixing being lumped into an eddy diffusivity, the outcome of a long-term horizontal dispersion coefficient can be qualitatively and quantitatively robust. Empiricism in the modelling of ν or the use of second-order turbulence models is likewise beyond the scope of the present paper (Hutton, Smith & Hickmott 1987).

The forced horizontal velocity components u, v are represented in terms of infinite series of free velocity modes:

$$u = \sum_{m=0}^{\infty} u^{(m)}(x, y, t)\Phi^{(m)}(\sigma), \quad v = \sum_{m=0}^{\infty} v^{(m)}(x, y, t)\Phi^{(m)}(\sigma)\ . \tag{2.3a,b}$$

In the context of the work of Svendsen & Putrevu (1994), u would be the off-shore under-tow and v the longshore current. The primary flow ($u^{(0)}$, $v^{(0)}$) will be associated with the zero modes and the secondary flow with the smaller-amplitude higher modes. However, a non-zero presence of the higher modes is found to be vital for there to be horizontal dispersion of the velocity.

When there is non-zero surface shear stress, σ-differentiation of the representations (2.3a,b) violates the convergence. The same conceptual difficulty arises in 'shallow water' models when vertically uniform velocities are used to model flows with no-slip at the bed and non-zero surface shear stress. As in traditional Galerkin methods, integration by parts with respect to σ will be used to avoid any need for direct representation of velocity derivatives. Hence, surface shear stress is accommodated in the amplitudes $u^{(m)}, v^{(m)}$ rather than by modifying the choice of modes $\Phi^{(m)}(\sigma)$.

The free modes $\Phi^{(m)}(\sigma)$, for the decay of unforced motion in the water, satisfy the eigenvalue problem (Part 1, equation (3.1); Part 2, (2.6)):

$$\frac{d}{d\sigma}\left(\hat{\nu}\frac{d\Phi^{(m)}}{d\sigma}\right) + \mu^{(m)}\Phi^{(m)} = 0, \tag{2.4a}$$

with

$$\Phi^{(m)} = 0 \quad \text{on} \quad \sigma = 0, \quad \hat{\nu}\frac{d\Phi^{(m)}}{d\sigma} = 0 \quad \text{on} \quad \sigma = 1, \tag{2.4b,c}$$

$$\int_0^1 \Phi^{(m)2} d\sigma = 1\ , \qquad \mu^{(m)} = \int_0^1 \hat{\nu}\left(\frac{d\Phi^{(m)}}{d\sigma}\right)^2 d\sigma, \tag{2.4d,e}$$

$$\int_0^1 \Phi^{(m)}\Phi^{(n)} d\sigma = 0 \quad \text{for} \quad m \neq n\ , \quad \text{and} \quad 0 < \mu^{(0)} < \mu^{(1)} < \dots \tag{2.4f,g}$$

The boundary conditions (2.4b,c) for the free modes correspond to zero slip at the bed and zero stress at the water surface. If any forcing were to vanish, the successive amplitudes $u^{(m)}, v^{(m)}$ would decay in time at the increasing rates $\mu^{(m)}N/H^2$.

We recall (Phillips 1957; Davies 1987; Part 1, (3.2c)) that the use of topography-following sigma-coordinates does not totally eliminate vertical velocities. The auxiliary modes needed to represent the vertical velocity involve σ-integrals:

$$\omega^{(m)}(\sigma) = \sigma \int_\sigma^1 \Phi^{(m)} \mathrm{d}\sigma' - (1-\sigma) \int_0^\sigma \Phi^{(m)} \mathrm{d}\sigma'. \tag{2.5}$$

In a multi-mode interpretation of shear dispersion for solute concentrations (Maron 1978; Yu & Chang 1994), it is the slow response and large amplitude of the $m = 0$ mode that distinguishes it. In a frame of reference moving at the appropriate speed, small systematic terms in the equation for the zero concentration mode can accumulate to large consequences. The classical example is the Taylor (1953) shear dispersion mechanism. By contrast, for the higher modes the response to small forcing terms is smaller, more immediate and easier to calculate.

3. Separation between the zero and higher modes

For laminar flows there is only a factor 4:9 disparity between the decay rates $\mu^{(0)}$ and $\mu^{(1)}$ of the zero and first velocity modes (Part 1, (5.3b)). Hence a Taylor dispersion model for velocity would not be justifiable. However, for turbulent open-channel flows the velocity modes do exhibit a more suitable 1:12 separation in response rates between the $m = 0$ and higher modes (Part 2, (2.6d)). We introduce a small parameter ε to characterize both the departure from unity of $\Phi^{(0)}$ and the small value of $\mu^{(0)}$:

$$\Phi^{(0)} = 1 + \varepsilon\Phi_1^{(0)}(\sigma) + \varepsilon^2\Phi_2^{(0)}(\sigma) + \dots\ , \quad \mu^{(0)} = \varepsilon\mu_1^{(0)} + \dots. \tag{3.1a,b}$$

For the higher modes the zero approximation $\Phi_0^{(m)}(\sigma)$, $\mu_0^{(m)}$ are the solute modes (Maron 1978) and satisfy a zero-flux bed condition. To satisfy the no-slip boundary condition for velocities (2.4b) the correction terms $\Phi_j^{(m)}$ need to become nearly singular at the bed $\sigma = 0$ (i.e. near the bed the eddy viscosity becomes exceedingly small). Svendsen & Putrevu (1994) allow instead for a slip velocity at the bed.

If the series (3.1) is substituted into equations (2.4a,d,e) and (2.5) then successive powers of ε yield numerous results. Those results required for the subsequent calculations are

$$\frac{\mathrm{d}\Phi_1^{(0)}}{\mathrm{d}\sigma} = \mu_1^{(0)}\frac{(1-\sigma)}{\hat{v}}\ , \quad \int_0^1 \Phi_0^{(m)}\mathrm{d}\sigma = 0\ , \quad \varepsilon\mu_1^{(0)}\int_0^1 \frac{(1-\sigma)^2}{\hat{v}}\mathrm{d}\sigma = 1 + \dots, \tag{3.2a,b,c}$$

$$\int_0^1 \omega_0^{(m)}\frac{\mathrm{d}\Phi_0^{(m)}}{\mathrm{d}\sigma}\mathrm{d}\sigma = 1 \quad (m \neq 0)\ , \quad \int_0^1 \Phi_1^{(m)}\mathrm{d}\sigma = \frac{\mu_1^{(0)}}{\mu_0^{(m)}}\Phi_0^{(m)}(0). \tag{3.2d,e}$$

The ε-factor in (3.2c) counterbalances the near singularity of the integral near the bed.

In the representation (2.2), perturbing the mismatch v' for fixed v is equivalent to perturbing the choice of the reference shape $\hat{v}(\sigma)$. It follows from equation (3.2c) that a small perturbation does not influence the decay rate $\varepsilon\mu_1^{(0)}$ for the zero velocity mode if

$$\int_0^1 v'\left(\frac{1-\sigma}{\hat{v}}\right)^2 \mathrm{d}\sigma = 0. \tag{3.3}$$

Mismatch near the bed is given much more weight than mismatch near the surface.

For bed-generated turbulence the classical von Kármán shape for the eddy viscosity, as used by Elder (1959) and in Part 2, is

$$\hat{v} = (1-\sigma)(\sigma + \exp(-\varepsilon^{-1})) , \quad \int_0^1 \frac{v'}{(\sigma + \exp(-\varepsilon^{-1}))^2} \mathrm{d}\sigma = 0, \tag{3.4a, b}$$

$$\Phi_1^{(0)} = 1 + \ln(\sigma + \exp(-\varepsilon^{-1})) , \quad \mu_1^{(0)} = 1, \tag{3.4c, d}$$

$$\Phi_0^{(m)} = (2m+1)^{1/2} P_m(2\sigma - 1) , \quad \mu_0^{(m)} = m(m+1), \tag{3.4e, f}$$

where P_m are Legendre polynomials. For a friction velocity to bulk velocity ratio of 1:15 and a von Kármán constant 0.4, we have $\varepsilon = 1/6$ and the ratio $\varepsilon\mu_1^{(0)} : \mu_0^{(1)}$ is 1:12.

For surface-generated turbulence an alternative exactly solvable model is

$$\hat{v} = (1-\sigma/2)(\sigma + \exp(-\varepsilon^{-1})) , \quad \int_0^1 \frac{v'}{(\sigma + \exp(-\varepsilon^{-1}))^2} \left(\frac{1-\sigma}{1-\sigma/2}\right)^2 \mathrm{d}\sigma = 0, \tag{3.5a, b}$$

$$\Phi_1^{(0)} = 2 - 2\ln 2 + \ln(\sigma + \exp(-\varepsilon^{-1})) + \ln(2-\sigma) , \quad \mu_1^{(0)} = 1, \tag{3.5c, d}$$

$$\Phi_0^{(m)} = (4m+1)^{1/2} P_{2m}(\sigma - 1), \quad \mu_0^{(m)} = m(2m+1). \tag{3.5e, f}$$

In Part 2 (§6 and figure 3) these two solvable models (3.4), (3.5) are used to demonstrate that the constraint (3.3) makes the modelling robust with respect to any v' mismatch. In the present calculations, the use of modes enables us to use equations derived in Parts 1 and 2 to facilitate the derivation of velocity dispersion formulae which do not involve the higher modes. Thus, there is no requirement that the modes be known explicitly, only that $\hat{v}$ has a shape appropriate to the flow being studied and that the zero-mode decay rate $\varepsilon\mu_1^{(0)}$ is accurately known.

4. Zero-mode equations

With the ε-expansions (3.1), (3.2), the leading-order terms in the vertically integrated mass conservation equation (Part 2, (3.1a)) take the familiar shallow water form

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x}(Hu^{(0)}) + \frac{\partial}{\partial y}(Hv^{(0)}) = H \int_0^1 Q \mathrm{d}\sigma. \tag{4.1}$$

$Q(x, y, \sigma, t)$ is the volumetric discharge rate for any sources of water. Correct to order ε^2, the velocity components $u^{(0)}, v^{(0)}$ are the same as the vertically averaged velocity.

Correct to leading order in ε, the $\Phi^{(0)}$ component of the horizontal momentum equations (Part 1, (3.4a,b); Part 2, (3.3a,b)) can be written

$$\frac{\partial}{\partial t}(Hu^{(0)}) + \frac{\partial}{\partial x}(Hu^{(0)2}) + \frac{\partial}{\partial y}(Hv^{(0)}u^{(0)}) + \sum_{m=1}^{\infty} \left\{ \frac{\partial}{\partial x}(Hu^{(m)2}) + \frac{\partial}{\partial y}(Hv^{(m)}u^{(m)}) \right\}$$

$$-Hfv^{(0)} + \varepsilon\mu_1^{(0)} \frac{N}{H} u^{(0)} = -H\left\{ \frac{1}{\rho_0}\frac{\partial P}{\partial x} + g\frac{\partial \zeta}{\partial x} \right\} + \frac{\tau_1}{\rho_0} + H\int_0^1 F_1 \mathrm{d}\sigma, \tag{4.2a}$$

$$\frac{\partial}{\partial t}(Hv^{(0)}) + \frac{\partial}{\partial x}(Hu^{(0)}v^{(0)}) + \frac{\partial}{\partial y}(Hv^{(0)2}) + \sum_{m=1}^{\infty} \left\{ \frac{\partial}{\partial x}(Hu^{(m)}v^{(m)}) + \frac{\partial}{\partial y}(Hv^{(m)2}) \right\}$$

$$+Hfu^{(0)} + \varepsilon\mu_1^{(0)} \frac{N}{H} v^{(0)} = -H\left\{ \frac{1}{\rho_0}\frac{\partial P}{\partial y} + g\frac{\partial \zeta}{\partial y} \right\} + \frac{\tau_2}{\rho_0} + H\int_0^1 F_2 \mathrm{d}\sigma. \tag{4.2b}$$

Here f is the Coriolis frequency, P atmospheric pressure, ρ_0 reference water density, g gravitational acceleration and $(\tau_1,\ \tau_2)$ are surface wind stresses. The additional forcing terms $F_1,\ F_2$ are defined

$$F_1 = M_1 - Q\sum_{m=0}^{\infty} u^{(m)}\Phi_0^{(m)} - \frac{gH}{\rho_0}\int_\sigma^1 \frac{\partial \rho'}{\partial x}\mathrm{d}\sigma' - g\frac{\partial H}{\partial x}\int_\sigma^1 (1-\sigma')\frac{\partial \rho'}{\partial \sigma'}\mathrm{d}\sigma', \qquad (4.3a)$$

$$F_2 = M_2 - Q\sum_{m=0}^{\infty} v^{(m)}\Phi_0^{(m)} - \frac{gH}{\rho_0}\int_\sigma^1 \frac{\partial \rho'}{\partial y}\mathrm{d}\sigma' - g\frac{\partial H}{\partial y}\int_\sigma^1 (1-\sigma')\frac{\partial \rho'}{\partial \sigma'}\mathrm{d}\sigma'. \qquad (4.3b)$$

The body forces $(M_1,\ M_2)$ can result from momentum of any water sources and (if wave motions have been time-filtered out) from gradients of the wave-related radiation stress (Longuet-Higgins & Stewart 1964). The density changes ρ' may be linked to solute concentrations (heat or salt). The $(u^{(m)},\ v^{(m)})$ summations in equations (4.2*a*,*b*) represent the effect upon the primary flow $(u^{(0)},\ v^{(0)})$ of any secondary flow. The neglect of the higher-mode summations results in the conventional shallow water momentum equations without any horizontal eddy viscosity terms.

At first sight it might appear that by virtue of the ε multiplier, the decay term in equations (4.2*a*, *b*) can be neglected. However, the dominant zero-mode nonlinearity can be removed simply by the expedient of using a frame of reference moving at the zero mode velocity $(u^{(0)},\ v^{(0)})$. If in such a frame the forcing terms vary sufficiently slowly (i.e. on a longitudinal length scale comparable with or longer than the decay distance $225H$) then, despite the ε multiplier, the decay terms do need to be retained.

In the context of solute dilution, Taylor(1953) recognized the need to use a moving frame in order to focus attention upon the small dispersion terms. Maron (1978) and Yu & Chang (1994) showed how the shear dispersion coefficient for solutes could be calculated in such a moving frame from the higher-mode summation terms. The next two sections do likewise for momentum.

5. Simplified equations for the higher modes

The full equations (Part 1, (3.4*a*,*b*); Part 2, (3.3*c*,*d*)) for $u^{(m)}$, $v^{(m)}$ are formidably complicated. However, the integrals (3.2*a*–*e*) and the smallness of $u^{(m)}$, $v^{(m)}$ relative to $u^{(0)}$, $v^{(0)}$ allows us to consider the simplified equations:

$$\frac{\mathrm{D}u^{(m)}}{\mathrm{D}t} + u^{(m)}\left\{\mu_0^{(m)}\frac{N}{H^2} - \frac{1}{2H}\frac{\mathrm{D}H}{\mathrm{D}t} + \frac{3}{2}\int_0^1 Q\mathrm{d}\sigma + \frac{1}{2}\frac{\partial u^{(0)}}{\partial x} - \frac{1}{2}\frac{\partial v^{(0)}}{\partial y}\right\} + v^{(m)}\left\{\frac{\partial u^{(0)}}{\partial y} - f\right\} = F_1^{(m)}, \qquad (5.1a)$$

$$\frac{\mathrm{D}v^{(m)}}{\mathrm{D}t} + v^{(m)}\left\{\mu_0^{(m)}\frac{N}{H^2} - \frac{1}{2H}\frac{\mathrm{D}H}{\mathrm{D}t} + \frac{3}{2}\int_0^1 Q\mathrm{d}\sigma - \frac{1}{2}\frac{\partial u^{(0)}}{\partial x} + \frac{1}{2}\frac{\partial v^{(0)}}{\partial y}\right\} + u^{(m)}\left\{\frac{\partial v^{(0)}}{\partial x} + f\right\} = F_2^{(m)}, \qquad (5.1b)$$

where

$$F_1^{(m)} = -\epsilon \frac{\mu_1^{(0)}}{\mu_0^{(m)}} \Phi_0^{(m)}(0) \left\{ \frac{1}{\rho_0} \frac{\partial P}{\partial x} + g \frac{\partial \zeta}{\partial x} \right\} + \frac{\tau_1}{H \rho_0} \Phi_0^{(m)}(1) + \int_0^1 \Phi_0^{(m)} F_1 \mathrm{d}\sigma, \quad (5.1c)$$

and

$$F_2^{(m)} = -\epsilon \frac{\mu_1^{(0)}}{\mu_0^{(m)}} \Phi_0^{(m)}(0) \left\{ \frac{1}{\rho_0} \frac{\partial P}{\partial y} + g \frac{\partial \zeta}{\partial y} \right\} + \frac{\tau_2}{H \rho_0} \Phi_0^{(m)}(1) + \int_0^1 \Phi_0^{(m)} F_2 \mathrm{d}\sigma. \quad (5.1d)$$

Here the advected derivative $\mathrm{D}/\mathrm{D}t$ is defined in terms of the zero-mode flow:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u^{(0)} \frac{\partial}{\partial x} + v^{(0)} \frac{\partial}{\partial y}. \quad (5.2)$$

If the forcing were to vanish, the amplitudes $u^{(m)}, v^{(m)}$ would decay in time at the rates $\mu_0^{(m)} N/H^2$ (at least 12 times more rapidly than the zero mode, on horizontal distances of order $20H$). In the moving frame of reference when the forcing varies slowly, the amplitudes $u^{(m)}, v^{(m)}$ are correspondingly much smaller than $u^{(0)}$, $v^{(0)}$.

The pressure gradient, surface wind stresses, momentum from water souces, wave-related radiation stress, and density changes all contribute to the forcing ($F_1^{(m)}$, $F_2^{(m)}$) for the higher modes. The relative sizes of the different driving forces in equations (5.1c,d) may be such that the pressure gradient contributions may be significant (despite the ε factor). For example, in the work of Svendsen & Putrevu (1994) the other terms are zero: there is no wind stress and the wave-related forcing F_1 is independent of σ, so has zero $\Phi_0^{(m)}$-weighted integral.

In equations (5.1a,b) horizontal derivatives of the zero-mode velocity components $u^{(0)}, v^{(0)}$ have been written in a symmetric form by the use of the mass conservation equation (4.1). Further symmetrization could be achieved by combining the Coriolis and vertical vortical components of angular velocity

$$\hat{f} = f + \tfrac{1}{2} \left\{ \frac{\partial v^{(0)}}{\partial x} - \frac{\partial u^{(0)}}{\partial y} \right\}. \quad (5.3)$$

However, attention will be restricted to tidal or higher-frequency flows, in which the Coriolis effect upon the secondary flow is small and the introduction of $\hat{f}$ is not necessary.

With the $\mathrm{D}/\mathrm{D}t$ terms neglected, equations (5.1a,b) become simultaneous linear equations for $u^{(m)}$, $v^{(m)}$. Away from discharges (i.e. neglecting Q), when the horizontal velocity gradients and the Coriolis frequency f are small relative to N/H^2, the solutions are

$$u^{(m)} = u_E^{(m)} - \tfrac{1}{2} \left\{ \frac{\partial u^{(0)}}{\partial x} - \frac{\partial v^{(0)}}{\partial y} \right\} \frac{H^2 u_E^{(m)}}{\mu_0^{(m)} N} - \left\{ \frac{\partial u^{(0)}}{\partial y} - f \right\} \frac{H^2 v_E^{(m)}}{\mu_0^{(m)} N} + \ldots, \quad (5.4a)$$

$$v^{(m)} = v_E^{(m)} + \tfrac{1}{2} \left\{ \frac{\partial u^{(0)}}{\partial x} - \frac{\partial v^{(0)}}{\partial y} \right\} \frac{H^2 v_E^{(m)}}{\mu_0^{(m)} N} - \left\{ \frac{\partial v^{(0)}}{\partial x} + f \right\} \frac{H^2 u_E^{(m)}}{\mu_0^{(m)} N} + \ldots, \quad (5.4b)$$

$$u_E^{(m)} = \frac{H^2 F_1^{(m)}}{\mu_0^{(m)} N}, \quad v_E^{(m)} = \frac{H^2 F_2^{(m)}}{\mu_0^{(m)} N}. \quad (5.4c,d)$$

Here $u_E^{(m)}$, $v_E^{(m)}$ are equilibrium approximations in the moving frame of reference for the directly forced secondary flow. The Coriolis effect and the horizontal straining of the primary flow perturb that basic secondary flow (tidal excursions and horizontal scales assumed to be of order $225H$).

6. Dispersion equations for the velocity components

In imitating solute dispersion calculations (Maron 1978; Yu & Chang 1994) the crucial terms in the momentum equations (4.2a, b) are the quadratic summation terms involving the higher modes $u^{(m)}$, $v^{(m)}$. The approximations (5.4a, b) lead to the expressions

$$\sum_{m=1}^{\infty} u^{(m)2} = R_{11} - D_{11}\left\{\frac{\partial u^{(0)}}{\partial x} - \frac{\partial v^{(0)}}{\partial y}\right\} - 2D_{12}\left\{\frac{\partial u^{(0)}}{\partial y} - f\right\} + \ldots, \tag{6.1a}$$

$$\sum_{m=1}^{\infty} u^{(m)} v^{(m)} = R_{12} - D_{11}\left\{\frac{\partial v^{(0)}}{\partial x} + f\right\} - D_{22}\left\{\frac{\partial u^{(0)}}{\partial y} - f\right\} + \ldots, \tag{6.1b}$$

$$\sum_{m=1}^{\infty} v^{(m)2} = R_{22} + D_{22}\left\{\frac{\partial u^{(0)}}{\partial x} - \frac{\partial v^{(0)}}{\partial y}\right\} - 2D_{12}\left\{\frac{\partial v^{(0)}}{\partial x} + f\right\} + \ldots . \tag{6.1c}$$

where the momentum fluxes R_{ij} and effective horizontal viscosities D_{ij} are given by

$$R_{11} = \sum_{m=1}^{\infty} u_E^{(m)2}, \quad R_{12} = \sum_{m=1}^{\infty} u_E^{(m)} v_E^{(m)}, \quad R_{22} = \sum_{m=1}^{\infty} v_E^{(m)2}, \tag{6.2}$$

$$D_{11} = \frac{H^2}{N}\sum_{m=1}^{\infty}\frac{u_E^{(m)2}}{\mu_0^{(m)}}, \quad D_{12} = \frac{H^2}{N}\sum_{m=1}^{\infty}\frac{u_E^{(m)} v_E^{(m)}}{\mu_0^{(m)}}, \quad D_{22} = \frac{H^2}{N}\sum_{m=1}^{\infty}\frac{v_E^{(m)2}}{\mu_0^{(m)}}. \tag{6.3}$$

The notation R_{ij} is reminiscent of the notation S_{ij} used by Longuet-Higgins & Stewart (1964) for the physically similar time-averaged momentum fluxes associated with the short waves. The stronger the equilibrium secondary flow in a particular direction, the larger the effective horizontal viscosity in that direction.

To the order of the approximations (5.4a, b) the effective horizontal viscosities D_{ij} are equal to the shear dispersion coefficients for solutes (Maron 1978, equation 33). Thus, to evaluate D_{ij} direct use can be made of previous investigations of shallow water solute dispersion such as Elder (1959) and Part 2. For solutes, horizontal shear dispersion coefficients are many times larger than vertical turbulent eddy diffusivities. Thus, effective horizontal viscosities are equally large (Svendsen & Putrevu 1994).

The corresponding formulation of the zero-mode momentum equations (4.2a, b), with mixed derivative terms written symmetrically, is

$$H\left(\frac{\mathrm{D}u^{(0)}}{\mathrm{D}t} + \varepsilon\mu_1^{(0)}\frac{N}{H^2}u^{(0)} - fv^{(0)}\right) - \frac{\partial}{\partial x}\left(HD_{11}\frac{\partial u^{(0)}}{\partial x} + HD_{12}\frac{\partial u^{(0)}}{\partial y}\right)$$
$$-\frac{\partial}{\partial y}\left(HD_{12}\frac{\partial u^{(0)}}{\partial x} + HD_{22}\frac{\partial u^{(0)}}{\partial y}\right)$$
$$= -H\left\{\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g\frac{\partial \zeta}{\partial x}\right\} + \frac{\tau_1}{\rho_0} + H\int_0^1 F_1\,\mathrm{d}\sigma - \frac{\partial}{\partial x}(HR_{11}) - \frac{\partial}{\partial y}(HR_{12}) + \ldots, \tag{6.4a}$$

$$H\left(\frac{\mathrm{D}v^{(0)}}{\mathrm{D}t}+\varepsilon\mu_1^{(0)}\frac{N}{H^2}v^{(0)}+fv^{(0)}\right)-\frac{\partial}{\partial x}\left(HD_{11}\frac{\partial v^{(0)}}{\partial x}+HD_{12}\frac{\partial v^{(0)}}{\partial y}\right)$$
$$-\frac{\partial}{\partial y}\left(HD_{12}\frac{\partial v^{(0)}}{\partial x}+HD_{22}\frac{\partial v^{(0)}}{\partial y}\right)$$
$$=-H\left\{\frac{1}{\rho_0}\frac{\partial P}{\partial y}+g\frac{\partial \zeta}{\partial y}\right\}+\frac{\tau_2}{\rho_0}+H\int_0^1 F_2\mathrm{d}\sigma-\frac{\partial}{\partial x}(HR_{12})-\frac{\partial}{\partial y}(HR_{22})+\ldots . \quad (6.4b)$$

Thus, in accord with the work of Svendsen & Putrevu (1994, equation 2.26), there is shear dispersion of the velocity. The D_{ij} terms quantify the feedback from horizontal straining of the primary flow via the perturbed secondary flow. The R_{ij} terms quantify the re-distribution of momentum by the equilibrium secondary flow and are written as right-hand-side forcing terms. The dots signify terms which are smaller than similar retained terms (e.g horizontal derivatives of HfD_{ij} neglected relative to $Hfu^{(0)}$ or $Hfv^{(0)}$).

7. Integral formulae for the shear dispersion coefficients

We define equilibrium approximations for the directly forced secondary flow:

$$u'=\sum_{m=1}^{\infty}u_E^{(m)}\Phi_0^{(m)}(\sigma),\quad v'=\sum_{m=1}^{\infty}v_E^{(m)}\Phi_0^{(m)}(\sigma), \quad (7.1a,b)$$

where the higher-mode amplitudes $u_E^{(m)}$, $v_E^{(m)}$ are defined in equation (5.4*c*,*d*). If in the full momentum equations (Part 1, (2.9*a*,*b*)) we retain just those terms corresponding to the approximations (5.4*c*,*d*) and allow for the removal of the zero mode, then we arrive at the greatly simplified equations and boundary conditions:

$$-\frac{N}{H^2}\frac{\partial}{\partial\sigma}\left(\hat{\nu}\frac{\partial u'}{\partial\sigma}\right)=F_1-\int_0^1 F_1\mathrm{d}\sigma'-\frac{\tau_1}{H\rho_0}+\epsilon\mu_1^{(0)}\left\{\frac{1}{\rho_0}\frac{\partial P}{\partial x}+g\frac{\partial\zeta}{\partial x}\right\}, \quad (7.2a)$$

$$-\frac{N}{H^2}\frac{\partial}{\partial\sigma}\left(\hat{\nu}\frac{\partial v'}{\partial\sigma}\right)=F_2-\int_0^1 F_2\mathrm{d}\sigma'-\frac{\tau_2}{H\rho_0}+\epsilon\mu_1^{(0)}\left\{\frac{1}{\rho_0}\frac{\partial P}{\partial y}+g\frac{\partial\zeta}{\partial y}\right\}, \quad (7.2b)$$

$$\frac{N}{H^2}\hat{\nu}\frac{\partial u'}{\partial\sigma}\to\epsilon\mu_1^{(0)}\left\{\frac{1}{\rho_0}\frac{\partial P}{\partial x}+g\frac{\partial\zeta}{\partial x}\right\}\ \text{and}\ \frac{N}{H^2}\hat{\nu}\frac{\partial v'}{\partial\sigma}\to\epsilon\mu_1^{(0)}\left\{\frac{1}{\rho_0}\frac{\partial P}{\partial y}+g\frac{\partial\zeta}{\partial y}\right\}\ \text{as}\ \sigma\to 0, \quad (7.2c,d)$$

$$\frac{N}{H^2}\hat{\nu}\frac{\partial u'}{\partial\sigma}\to\frac{\tau_1}{H\rho_0}\quad\text{and}\quad\frac{N}{H^2}\hat{\nu}\frac{\partial v'}{\partial\sigma}\to\frac{\tau_2}{H\rho_0}\quad\text{as}\quad\sigma\to 1, \quad (7.2e,f)$$

with

$$\int_0^1 u'\mathrm{d}\sigma=0\ ,\quad \int_0^1 v'\mathrm{d}\sigma=0. \quad (7.2g,h)$$

In the frame of reference moving with the dominant primary flow ($u^{(0)}$, $v^{(0)}$) the time response for the directly forced secondary flow is approximated as being immediate. It deserves comment that the boundary conditions (7.2c,d) at the bed do admit slip . So, it may not be a coincidence that the discovery by Svendsen & Putrevu (1994) of a shear dispersion mechanism for velocity was in the context of a flow model with slip.

The normalization (2.4d) and orthogonality (2.4f) of the modes $\Phi_0^{(m)}$ allows us to identify the summations (6.2) and (6.3) for the momentum fluxes and shear dispersion coefficients with the vertical integrals:

$$R_{11} = \int_0^1 u'^2 \mathrm{d}\sigma, \quad R_{12} = \int_0^1 u'v' \mathrm{d}\sigma, \quad R_{22} = \int_0^1 v'^2 \mathrm{d}\sigma, \tag{7.3a,b,c}$$

$$D_{11} = \frac{H^2}{N} \int_0^1 \left\{ \int_0^\sigma u' \mathrm{d}\sigma' \right\}^2 \frac{\mathrm{d}\sigma}{\hat{\nu}}, \quad D_{22} = \frac{H^2}{N} \int_0^1 \left\{ \int_0^\sigma v' \mathrm{d}\sigma' \right\}^2 \frac{\mathrm{d}\sigma}{\hat{\nu}}, \tag{7.4a,b}$$

$$D_{12} = \frac{H^2}{N} \int_0^1 \left\{ \int_0^\sigma u' \mathrm{d}\sigma' \right\} \left\{ \int_0^\sigma v' \mathrm{d}\sigma' \right\} \frac{\mathrm{d}\sigma}{\hat{\nu}}. \tag{7.4c}$$

There are further simplifications in equations (7.2), (7.3), (7.4) if we were to neglect the mismatch ν' and replace $N\hat{\nu}$ by the eddy viscosity ν.

In a different style of calculation (numerical or analytical), in which the secondary flow u', v' is calculated directly, the integral formulae (7.3), (7.4) provide a convenient alternative to the summations (6.2), (6.3). Equation (7.4*a*) is equivalent to one of the central results of Svendsen & Putrevu (1994, equation 2.22) for the effective longitudinal viscosity. Equations (7.4*b*,*c*) give the two-dimensional extension.

It is reasonable to suppose that an alternative derivation might be made of the zero-mode equations (6.4), the equilibrium secondary flow equations (7.2) and the dispersion integrals (7.3), (7.4) , without the need to invoke any but the zero mode. The zero velocity mode cannot be avoided because of the repeated appearances of the decay rate $\varepsilon\mu_1^{(0)}$.

8. Concluding remarks

In view of the complexity and generality of the physical processes modelled, the simplicity of the shear dispersion equations (6.4*a*,*b*) for the horizontal velocity components is striking. In particular, the shear dispersion tensor D_{ij} is the same for both velocity components as for solutes and there are no diffusive terms cross-linking the two velocity components $u^{(0)}$, $v^{(0)}$. Thus, established results (Elder 1959) for the horizontal solute dispersion tensor in shallow water flows also yield the effective horizontal viscosities.

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